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Bispectra of internal waves

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This note summarizes a detailed numerical computation of bispectra arising from weak nonlinear resonant interactions of internal waves whose energies are represented by the Garrett & Munk (1975) model spectrum. Two spectra are computed – the bispectrum of power and the auto-bispectrum of vertical displacement. These are chosen because the first is the most informative and the second is easy to observe. The numerical computations indicate that the level of the bispectral signal is much too low to be detected by any reasonable observational programme. Even more disturbing, bispectra of Eulerian variables are subject to a kinematic contamination causing a significant bispectral level which can easily be misinterpreted as a nonlinear interaction.

1. Introduction

Bispectra, the double Fourier decomposition of third-order moments, have been used in turbulence (Lii, Rosenblatt & Van Atta, 1976; Helland, Van Atta & Steger, 1977) and in ocean surface waves (Hasselmann, Munk & MacDonald 1963) to observe nonlinear coupling among wave components. The former computed the 'bispectrum of power' which indicates the rate of energy transfer into one component from interaction of two others; the latter successfully compared theoretical and observational bispectra.

McComas & Bretherton (1977) and McComas (1977) have shown that there are strong, scale-separated, resonant interactions occurring in the internal wave field. It was hoped that these interactions would produce a characteristic bispectral signal of sufficient strength to be verified by observation.

Section 2 presents a very brief description of the theoretical development for the numerical computation of Lagrangian bispectra. Computations using the Garrett & Munk spectral model demonstrate the futility of bispectral analysis for indicating ocean internal wave interactions. Much greater detail is available in McComas (1978).

Section 3 shows that Lagrangian and Eulerian bispectra can be different if the field is Gaussian to first order. The Eulerian bispectra are subject to a kinematic contamination similar to the familiar fine-structure contamination of variance spectra. We develop here a model for self-contamination for the quasi-Eulerian variables used by Neshyba & Sobey (1975; hereinafter referred to as NS) and reproduce the major features of their result. Combined with the computations of § 2 and further observational results (Briscoe & McComas 1979), we feel that NS's result cannot be interpreted as evidence of nonlinear interaction.

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2. Numerical computations

The cross-bispectrum of any stationary homogeneous random variables $u(\mathbf{x},t)$, $v(\mathbf{x},t)$ and $w(\mathbf{x},t)$ may be defined as

$$B_{uvw}(\boldsymbol{\kappa}',\omega',\mathbf{k}'',\omega'') = \frac{1}{(2\pi)^8}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \times T_{uvw}(\mathbf{r}',\tau',\mathbf{r}'',\tau'')\exp[-i\boldsymbol{\kappa}'.\mathbf{r}-i\omega'\tau'-i\boldsymbol{\kappa}''.\mathbf{r}''-i\omega''\tau'']\,d\mathbf{r}'d\mathbf{r}''d\tau'd\tau'' \quad (1)$$

where $\quad T_{uvw}(\mathbf{r}',\tau',\mathbf{r}'',\tau'') = \langle u(\mathbf{x},t)\,v(\mathbf{x}+\mathbf{r}',t+\tau')\,w(\mathbf{x}+\mathbf{r}'',t+\tau'')\rangle.$

The bispectrum is then the double Fourier transform of a third-order correlation. If u, v and w are independent Gaussian processes then this correlation is zero at all separations, and the bispectrum is zero at all $\boldsymbol{\kappa}$, ω. Any process which is not strictly stationary, homogeneous, or Gaussian will have a non-zero bispectrum. It is the departure from Gaussianity due to dynamic nonlinear coupling that is of interest here.

McComas (1978) has shown that for a weakly nonlinear field, whose amplitude can be expanded in powers of a small parameter and whose first order field is linear and Gaussian (the usual assumptions leading to weak resonant interaction), that any bispectrum of Lagrangian variables is given by the general form

$$B_{uvw}(\boldsymbol{\kappa}',\omega',\boldsymbol{\kappa}'',\omega'') = H^{-s'-s''-s}_{-\boldsymbol{\kappa}'-\boldsymbol{\kappa}''-\boldsymbol{\kappa}}\,\delta(\boldsymbol{\kappa}+\boldsymbol{\kappa}'+\boldsymbol{\kappa}'')\,\delta(s\Omega(\boldsymbol{\kappa})+s'\Omega(\boldsymbol{\kappa}')+s''\Omega(\boldsymbol{\kappa}'')) \times[sA(\boldsymbol{\kappa}')A(\boldsymbol{\kappa}'')+s'A(\boldsymbol{\kappa})A(\boldsymbol{\kappa}'')+s''A(\boldsymbol{\kappa})A(\boldsymbol{\kappa}')], \quad (2)$$

where H is a complicated complex interaction coefficient depending only on $\boldsymbol{\kappa},\boldsymbol{\kappa}',\boldsymbol{\kappa}''$ and on the choice of variables u, v, w; $s=\pm 1$, $\Omega(\boldsymbol{\kappa})$ is the dispersion relation, and $A(\boldsymbol{\kappa})$ is the wave action density. The form of (2) is that of the integrand in Hasselmann's (1966) formulation for the rate of change of the action density. In fact, the bispectrum of power, defined as that combination of variables which indicates the time rate of change of energy, i.e. the nonlinear terms in the energy equation, is precisely that integrand times the frequency

$$\begin{aligned}B_{\text{power}}(\boldsymbol{\kappa}',\omega',\boldsymbol{\kappa}'',\omega'') &= \omega\frac{\partial A(\boldsymbol{\kappa})}{\partial t} = \frac{\partial E(\boldsymbol{\kappa})}{\partial t}\\ &= \omega\left|D^{-s-s'-s''}_{-\boldsymbol{\kappa}-\boldsymbol{\kappa}'-\boldsymbol{\kappa}''}\right|^2\delta(\boldsymbol{\kappa}+\boldsymbol{\kappa}'+\boldsymbol{\kappa}'')\,\delta(s\Omega(\boldsymbol{\kappa})+s'\Omega(\boldsymbol{\kappa}')+s''\Omega(\boldsymbol{\kappa}''))\\ &\quad\times[sA(\boldsymbol{\kappa}')A(\boldsymbol{\kappa}'')+s'A(\boldsymbol{\kappa})A(\boldsymbol{\kappa}'')+s''A(\boldsymbol{\kappa})A(\boldsymbol{\kappa}')]. \quad (3)\end{aligned}$$

A more complete derivation of (2) and (3) is contained in McComas (1978).

Given some model for the action density spectrum, any bispectrum can be predicted by a numerical evaluation of these formulas and then compared to bispectra obtained from data.

The theoretical bispectrum (3) is easy to evaluate as only one triad contributes to each point in the eight-dimensional bispectrum. However, besides being awkward to display, such a bispectrum would require observations in all space dimensions, or two space dimensions and time, either of which is a considerable task. In order to compare the numerical results with observations, the number of arguments can be reduced by integrating the bispectrum over the other arguments. Because time series records are most readily available, the frequency bispectrum was chosen. Thus

$$B(\omega',\omega'') = \int_{-\infty}^{\infty}\cdots\int B(\boldsymbol{\kappa}',\omega',\boldsymbol{\kappa}'',\omega'')\,\delta(\omega'-\Omega(\boldsymbol{\kappa}'))\,\Omega(\omega''-\Omega(\boldsymbol{\kappa}'')\,d\boldsymbol{\kappa}'d\boldsymbol{\kappa}'', \quad (4)$$

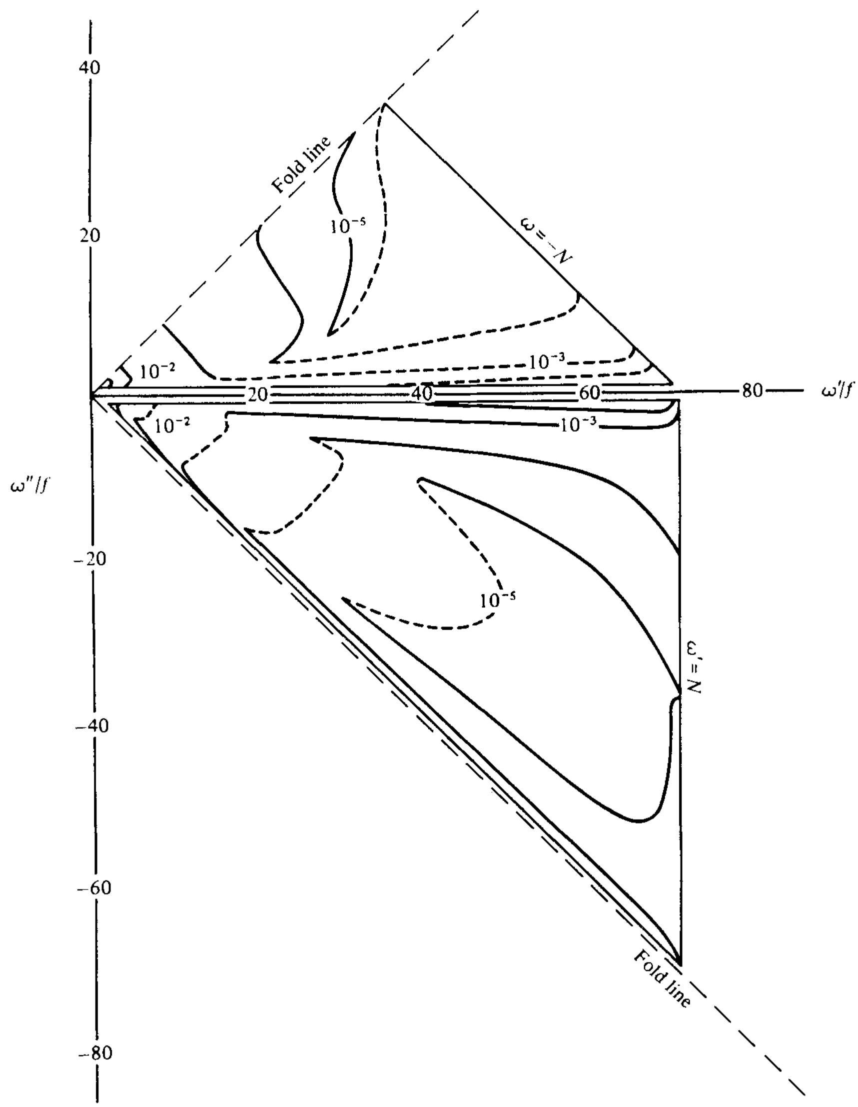

FIGURE 1. The frequency bispectrum of power contoured in powers of ten in $m^2 s^{-1}$. Dashed contours indicate negative values. Thin solid lines represent boundaries outside of which the bispectral level is zero because one of the frequencies is smaller than $f = 7 \times 10^{-5} s^{-1}$ or larger than $N = 5 \times 10^{-3} s^{-1}$.

where $B(\boldsymbol{\kappa}', \omega', \boldsymbol{\kappa}'', \omega'')$ is given by (2) or (3). Equation (4) is what has been numerically evaluated based on the Garrett & Munk (1975) spectrum. McComas (1978) discusses other projections.

The results are displayed in figure 1 for the bispectrum of power and in figure 3 for the auto-bispectrum of vertical displacements. It is not the purpose of this note to detail the features of these numerical results. We simply indicate that there is a ridge in both bispectra where one frequency is small. These can be identified with the dominant interaction mechanisms noted by McComas & Bretherton (1977).

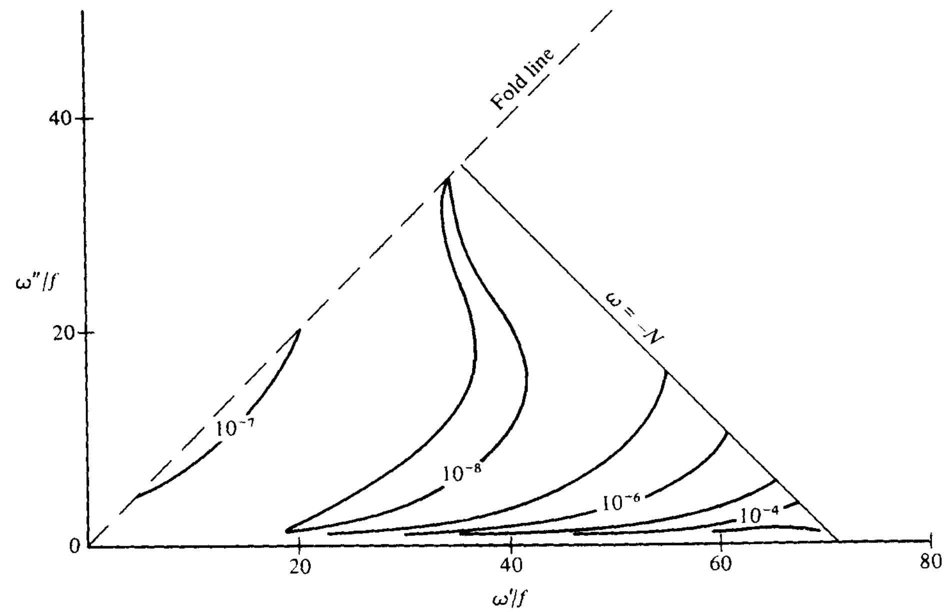

FIGURE 2. Contour plot of the squared bicoherence for the frequency bispectrum of power. Only the upper octant is shown.

As we hope to make comparisons of these theoretical bispectra to observed bispectra, we should inquire as to whether the bispectral levels are sufficiently high to be detected with statistical confidence in any reasonable experiment. Haubrich (1965) has shown that the required number of estimated degrees of freedom (e.d.o.f.) for the 95 % confidence level for the squared bicoherence is

$$\nu = 4/C^2(\omega', \omega''; \Delta\omega', \Delta\omega''), \tag{5}$$

where C is the bicoherence at ω', ω'' for a band-width $\Delta\omega', \Delta\omega''$. The bicoherence may be defined as

$$C_{uvw}(\omega', \omega''; \Delta\omega', \Delta\omega'') = \frac{|B_{uvw}(\omega', \omega'')\,\Delta\omega'\Delta\omega''|}{|P_u(\omega)\,\Delta\omega P_v(\omega')\,\Delta\omega' P_w(\omega'')\,\Delta\omega''|^{\frac{1}{2}}}, \tag{6}$$

where $P_u(\omega)$ is the variance spectrum of u. Thus, C is zero if u, v, w are independent Gaussian processes. However, in any finite sample size, C will be non-zero even for a Gaussian process. For any given bispectral level, the e.d.o.f. must exceed ν if the process is to be distinguished from a Gaussian process with 95 % confidence.

Figures 2 and 4 show plots of C^2 for a band-width equal to the inertial frequency f for the two computed bispectra. The smallest ν is of the order of 1000; since it involves waves with a period of approximately one day it indicates a 3 year experiment is required! Clearly, the detection of such a low bispectral level is not feasible.

Why should these interactions be so difficult to detect? There are several reasons. First, the interactions are not very strong. Consider an order-of-magnitude estimate for the squared bicoherence of power

$$C^2_{\text{power}}(\omega', \omega''; \Delta\omega', \Delta\omega'') \simeq \frac{|(\partial E/\partial t)\,(\omega; \omega', \omega'')|^2}{E(\omega)\,E(\omega')\,S(\omega'')}, \tag{7}$$

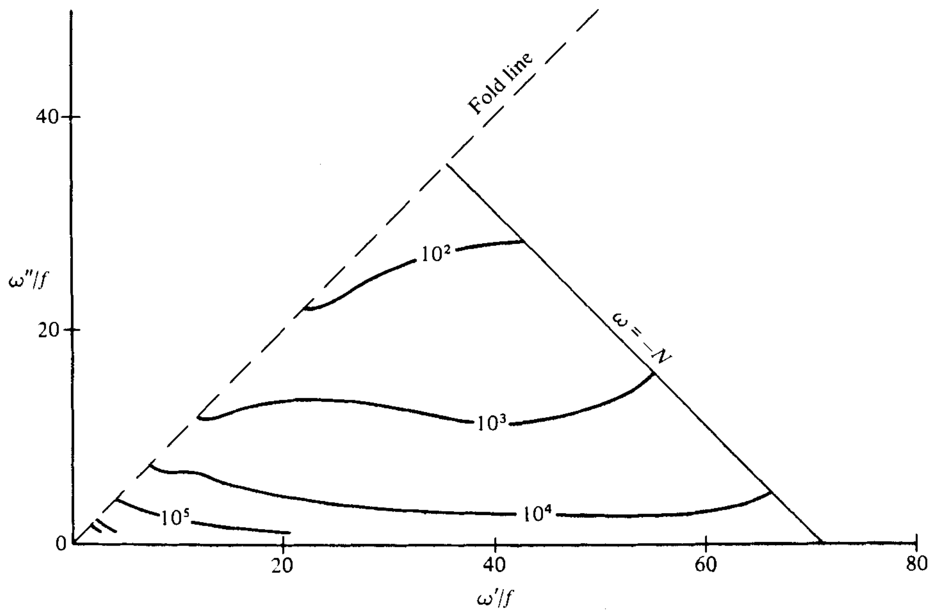

FIGURE 3. Contour plot of the frequency auto-bispectrum of Lagrangian vertical displacements in $m^3 s^2$.

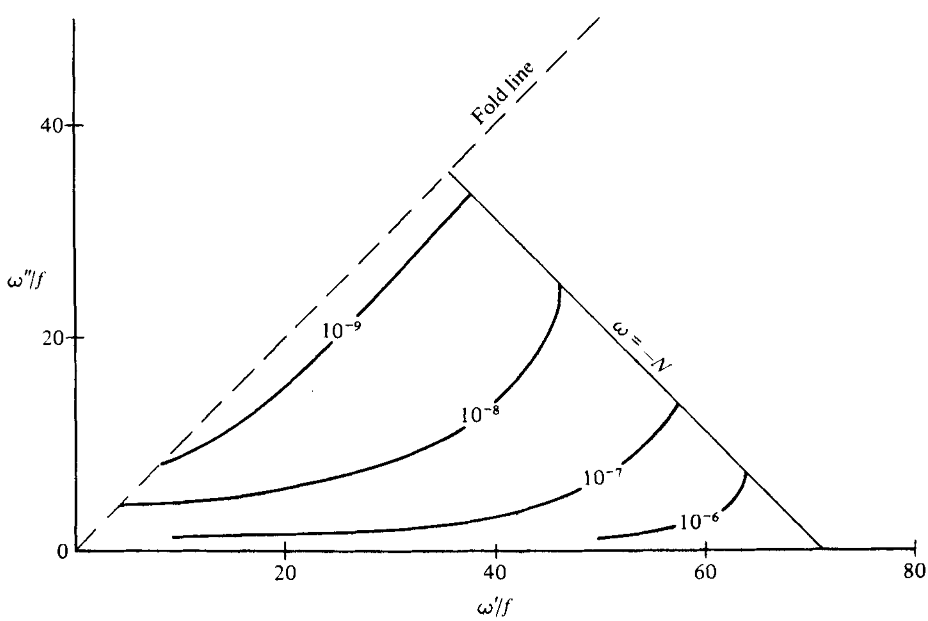

FIGURE 4. Contour plot of the squared bicoherence of the frequency auto-bispectrum of Lagrangian vertical displacements.

where E is the energy spectrum and S is the shear spectrum. If S is the total shear, T_N the buoyancy time scale, and T_{int} is the interaction time scale, then

$$C^2 \simeq \left(\frac{T_{\text{int}}}{T_N}\right)^{-2} \frac{E(\omega)}{E(\omega')} \frac{S}{S(\omega'')}. \tag{8}$$

Taking the strong ridge in figure 2 with $\omega' = 40f$, $\omega'' = 1{\cdot}5f$, $\Delta\omega = \Delta\omega' = \Delta\omega'' = 1f$ and evaluating (8) using the Garrett & Munk spectrum gives

$$O(\nu) = \left(\frac{T_{\text{int}}}{T_N}\right)^2 \tag{9}$$

such that for a several day experiment, the interaction time must be of the order of the buoyancy time, or equivalently, the period. This is a rapid, strong interaction!

Perhaps even more restrictive, the interaction must be due only to interactions with a particular ω' and ω''. However, there are many combinations of frequencies that sum to ω and all contribute to $E(\omega)$. Although a single wave component may be interacting with two other waves in a nonlinear and bispectrally-detectable way, that wave component is also simultaneously interacting with many other wave pairs. The low utility of bicoherence is a statistical dilemma arising from trying to use a three-wave idea in a many-wave random field. At second order, the concept of partial coherence (Bendat & Piersol 1971) is used to treat a similar difficulty; we have no similar theory of partial bicoherence.

Finally, for the particular case of internal waves, we note that large scale waves are energetic, but not very nonlinear, while the fairly nonlinear waves are not very energetic. Thus the bicoherence normalizes the nonlinearity of the small scales with the energy of the large scales, which results in a low bicoherence. (Internal wave frequency is independent of spatial scale.)

To summarize, numerical computations of bispectra of resonantly interacting oceanic internal waves indicate that bispectra from observed records should not be significantly different from zero.

3. Bispectral contamination

The previous section appears to contradict the statistically significant bispectrum of vertical displacements reported by NS. This section will demonstrate that Eulerian bispectra are subject to a kinematic contamination, such that even a linear, uncoupled field gives a non-zero bispectrum reproducing the characteristic features of the NS result. Together with the preceding section, we therefore suggest their result cannot be ascribed to nonlinear coupling.

Any Eulerian variable $Q_E(x,t)$ can be related to the Lagrangian variable $Q_L(x,t)$ through the Lagrangian displacement $\boldsymbol{\xi}_L(x,t)$ by

$$Q_L(\mathbf{x},t) = Q_E(\mathbf{x},t) - \boldsymbol{\xi}_L(\mathbf{x},t) . \nabla Q_L(\mathbf{x},t) + \dots \tag{10}$$

correct to second order (Phillips 1966). Third-order time correlations and frequency bispectra are then related by

$$\langle Q_E Q_E Q_E \rangle = \langle Q_L Q_L Q_L \rangle - \langle \boldsymbol{\xi}_L . \nabla Q_L Q_L Q_L \rangle - \langle Q_L \boldsymbol{\xi}_L . \nabla Q_L Q_L \rangle - \langle Q_L Q_L \boldsymbol{\xi}_L . \nabla Q_L \rangle + \dots . \tag{11}$$

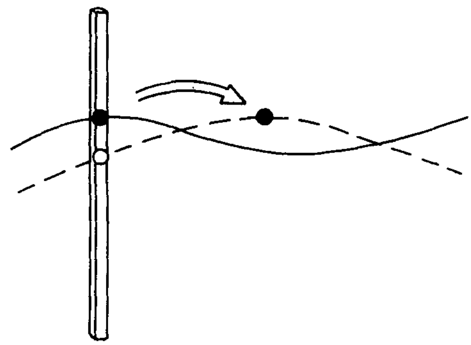

FIGURE 5. Sketch of constant temperature surface at $t = t_0$ (solid) and $t = t_0 + \Delta t$ (dashed). A particle (●) at the Eulerian line sensor (vertical bar) at time t_0 will be displaced away from the line sensor at $t_0 + \Delta t$. The line sensor will be measuring the depth position of the constant temperature surface for a different particle (○). Hence this measurement is not purely Lagrangian.

If the wave motion is of small amplitude, the third-order products are nearly equal – unless the Lagrangian statistics are Gaussian. Then the Lagrangian triple correlation is zero but the fourth-order terms are not and, hence, the Eulerian triple correlation is non-zero. These fourth-order terms can be thought of as a third-order correlation between Q_L, Q_L, and $\boldsymbol{\xi}_L.\nabla Q_L$, a 'false' signal which arises because gradients of Q_L are swept past the fixed sensor. This is analogous to the more familiar 'finestructure contamination' found in second-order spectra. Here it is more serious, as the real signal is zero and only the kinematic contamination gives any result.

This contamination is a possible explanation for the significant bispectrum of NS. Their vertical displacement time series were taken from encounters of undulating step-like layers on a vertical array of temperature sensors deployed from an ice island. Thus the measurement is approximately Eulerian in the horizontal, but neither Lagrangian nor Eulerian in the vertical figure (5). Using (1) and (11), and assuming the Lagrangian field is Gaussian so that fourth-order moments can be expressed as products of second moments, and additionally horizontal isotropy, then the NS bispectrum is given by

$$B_{\zeta\zeta\zeta}(\omega',\omega'') = 0 \quad \text{if} \quad \omega',\omega'',\omega \quad \text{all} \quad \neq 0$$
$$= \sum_{\omega_1}\left\langle \xi_i(\omega_1)\frac{\partial \xi_3(-\omega_1)}{\partial x_i}\right\rangle \langle \xi_3(\omega')\,\xi_3(-\omega')\rangle, \quad i = 1,2 \quad \text{if} \quad \omega = 0, \qquad (12)$$

and similarly if $\omega' = 0$ or $\omega'' = 0$. The first term in the brackets is the power of the 'false' vertical displacement due to the sweeping of horizontal gradients in vertical displacements past the fixed Eulerian sensor. Remember that this non-zero bispectral value arises even when the Lagrangian field is non-interacting (Gaussian). Furthermore (12) is non-zero precisely where NS found their strongest results, i.e. wherever one of the frequencies is in the lowest frequency bin. Neshyba has found (personal communication 1978) that these bispectral values are not significant if the *trend* is removed from the data. An evaluation of (12) using the Garrett & Munk (1975) model with a 10 % vertical asymmetry yields a bicoherence of about 0·5, several orders of magnitude higher than that indicated by the theory.

We are not suggesting that (12) precisely explains NS's result, for many assumptions have been made. However, in view of the large bicoherences possible from the kinematic contamination of Eulerian variables, the stark disagreement with theoretical prediction, and the lack of confirmation from further observations (Briscoe & McComas 1979), we conclude that NS does not reflect *dynamic* nonlinear coupling.

4. Conclusions

Numerical computations based on weakly nonlinear, resonant interaction theory indicate that observationally determined bispectra of oceanic internal waves should not differ significantly from zero. The only dissenting observation (NS) has been shown to be subject to a kinematic contamination that could account for that result.

The low utility of bispectra for oceanic internal waves is a result of the general insensitivity of bicoherence to weak nonlinearity, the promiscuity of the interactions (i.e. there are many triads contributing to each component), and the lack of strong interactions among the energy containing components of the internal wave field. We do not denegrate bispectral analysis for processes where nonlinearity between three strong components is suspected (e.g. internal tides) or where strong interaction in a peak is occurring (e.g. surface waves). However, for situations not in these categories, one must be prepared to take large amounts of data to achieve statistical significance; e.g. Helland *et al.* (1977) required 10^4 e.d.o.f. to achieve marginal significance in wind tunnel turbulence, a strongly nonlinear phenomenon!

Other than the indication that energetic scales are not strongly coupled by one dominant mechanism, we have so far learned little from bispectra of internal waves. In particular, the validity of weak resonant interaction has neither been confirmed nor denied.

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